The Imbedding Sum of a Graph

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Abstract

The automorphisms of a graph act naturally on its set of labeled imbeddings to produce its unlabeled imbeddings. The imbedding sum of a graph is a polynomial that contains useful information about a graph's labeled and unlabeled imbeddings. In particular, the polynomial enumerates the number of different ways the unlabeled imbeddings can be vertex colored and enumerates the labeled and unlabeled imbeddings by their symmetries.

1 Introduction

1.1 Motivation

Graphical enumeration is a well-established discipline that counts the number of graphs with a selected list of properties. These counting problems frequently arise in the structure of chemical compounds, in computer science and in combinatorial designs. In the article by Mull, Rieper, and White [8] a technique was developed for counting the number of different imbeddings a graph has on orientable surfaces. The result was extended to include imbeddings on nonorientable surfaces by Kwak and Lee [6]. For example, the complete graph on four vertices has three different imbeddings among the orientable surfaces (one on the plane and two on the torus) and eleven among all surfaces including the nonorientable cases. Similar results for complete bipartite graphs were found by Mull [7].

The purpose of this article is to show how these results can be extended to obtain information about graph imbeddings beyond just their quantity. We show, for example, that the three different orientable imbeddings of K_4 become five if two of the vertices are distinguished from the other two by coloring them black and the others white. We also show how the imbeddings are distributed according to the symmetries that they have.

1.2 Graph imbeddings

An *imbedding* of a graph G, considered to be a simplicial complex, into a closed orientable 2-manifold S, is a homeomorphism i of G into S, $i : G \to S$. If every component of $S - i(G)$ is a 2-cell (a homeomorph of an open disk), then i is a *2-cell* imbedding. If an orientation is provided for S, then i is an *oriented* imbedding. The orientable 2-manifolds considered here are the connected compact topological spaces each of which is characterized as a sphere with handles. Such a space is called an *orientable surface* and the number of handles is its *genus*. We regard two oriented imbeddings $i : G \to S$ and $j : G \to S$ as the same if there exists an orientation-preserving homeomorphism h of S onto S such that $h \circ i = j$. Henceforth, a 2-cell oriented imbedding is referred to as a *labeled imbedding*.

Each labeled imbedding of a graph can be described by the following well-known scheme (see Gross and Tucker [3]). In a small neighborhood of a vertex v we observe the counterclockwise cyclic order of the edges incident with v. If the graph has no loops or multiple edges, then we record the vertices adjacent to v in this order as the *rotation* at v. The *rotation system* is the vertex-indexed set of these rotations. If the vertex v is adjacent to degree(v) other vertices, then there are (degree(v)-1)! different rotations at v. The product over the vertex set of these numbers is the quantity of different labeled imbeddings.

For a given graph, many of its labeled imbeddings resemble one another. Formally, two labeled imbeddings $i : G \to S$ and $j : G \to S$ are *congruent* if there exists a graph automorphism $h : G \to G$ such that $i \circ h = j$. A congruence class is called an *unlabeled imbedding*. Informally, a labeled imbedding of a graph is a drawing of the graph on a surface where each vertex receives a label. In an unlabeled imbedding the vertex labels are omitted. We remark that the definitions of rotation scheme and of labeled and unlabeled imbeddings can be naturally extended to include graphs with multiple edges or loops and to directed graphs.

2 The Imbedding Sum of a Graph

Let G be a connected *simple* graph with vertex set V where a simple graph has no loops or multiple edges. The restriction to simple graphs enables us to more easily describe the automorphisms (they act on the vertices alone) and is easily circumvented by modeling a more general graph with a suitable simple one. The *neighborhood* $N(v)$ of the vertex v is the set of vertices adjacent to v. Each rotation at v is a cyclic permutation of the neighborhood of v, denoted $\rho_v : N(v) \to N(v)$. The rotation system is the indexed set $\rho = \{\rho_v\}_{v \in V}$. A *map* is a pair $M = (G, \rho)$ and provides an algebraic correspondence with the labeled imbeddings.

We denote the automorphism group of G by $\Gamma(G)$ (or simply Γ if G is understood) and define two maps $M = (G, \rho)$ and $M' = (G, \rho')$ to be *equivalent* if there exists an automorphism $\gamma \in \Gamma(G)$ such that $\gamma \rho_v \gamma^{-1} = \rho'_{\gamma(v)}$ for all vertices v. The graph automorphism γ can be interpreted as relabeling the vertices of G. Equivalent maps correspond with congruent labeled imbeddings. A *map automorphism* for $M = (G, \rho)$ is a graph automorphism giving M equivalent to itself. The set of map automorphisms for

$M=(G,\rho)$ form the map-automorphism subgroup $\Gamma_M(G)$ of $\Gamma(G)$. Thus, the graph-automorphism group $\Gamma(G)$ acts on the set of maps and $\Gamma_M(G)$ is the stabilizer of the map M under this action. We therefore have the following result (Biggs [1]).

Theorem 1. *The number of labeled imbeddings of the graph G in the unlabeled class containing the map M is the index of $\Gamma_M(G)$ in $\Gamma(G)$.*

The next three theorems, called the *counting theorems* are those derived in [8] and are to be used extensively in the applications to follow. For this reason they are reproduced here without proof. For each graph automorphism $\gamma \in \Gamma$ we let $F(\gamma)$ be the set of maps which are fixed by γ. That is, $M \in F(\gamma)$ if and only if M is γ-equivalent to itself. An application of Burnside's lemma yields the next result.

Theorem 2. *The number of different unlabeled imbeddings of a graph is $\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}|F(\gamma)|$.*

The cardinality of a fixed set is determined as follows. Each automorphism γ of the graph G does double duty. It acts on the set of vertices and the set of maps. If v is a vertex of G, then we let $l(v,\gamma)$ denote the length of the orbit which contains the vertex v under the action of γ. We define the *fixed set at v*, denoted $F_v(\gamma)$, to be the set of rotations at v which are fixed by γ under conjugation. If an automorphism γ fixes a rotation system ρ, then $\gamma\rho_v\gamma^{-1}=\rho_{\gamma(v)}$ for each vertex v. It follows that $\gamma^{l(v,\gamma)}\rho_v\gamma^{-l(v,\gamma)}=\rho_v$, or ρ_v is a member of the set $F_v(\gamma^{l(v,\gamma)})$. It is the cardinalities of these sets which determine the cardinality of $F(\gamma)$.

Theorem 3. *If $\gamma\in\Gamma$, then $|F(\gamma)|=\prod_{v\in S}\left|F_v\left(\gamma^{l(v,\gamma)}\right)\right|$, where the product extends over a complete set S of orbit representatives of γ acting on the vertex set V.*

A permutation is defined to be *d-regular* if each of its orbits has cardinality d. That is, in the disjoint cycle representation of the permutation, all the cycles have length d. The next result provides the number of rotations at a vertex which are fixed by the automorphism $\gamma^{l(v,\gamma)}$. We let ϕ denote the Euler function and recall that degree(v) denotes the cardinality of the neighborhood $N(v)$ of v.

Theorem 4.

$$\left|F_v\left(\gamma^{l(v,\gamma)}\right)\right|=\begin{cases}\phi(d)\left(\frac{\text{degree}(v)}{d}-1\right)!\,d^{\frac{\text{degree}(v)}{d}-1} & \textit{if } \gamma^{l(v,\gamma)} \textit{ is } d\textit{-regular on } N(v),\\ 0 & \textit{otherwise.}\end{cases}$$

We claim that there is much more information contained in the above counting theorems than previously reported. To obtain the additional information we supplement Theorem 2 with a related but more detailed result.

Each automorphism acting on the vertices of the graph carries with it a *cycle type* which records the number of orbits of a particular length. For example, $\gamma=(v_1v_2)(v_3v_4)(v_5v_6v_7)$ has two 2-cycles and a 3-cycle. The cycle type of any permutation of an n-set is encoded as a monomial in the indeterminates $s_1,s_2,\ldots,s_n$, where the exponent j_k of s_k in the monomial is the number of cycles of length k. The cycle type of the above permutation is encoded as $s_2^2s_3$. For simplicity we let s denote the function which assigns to a permutation the monomial that encodes its cycle type; that is, $s(\gamma)=\prod s_k^{j_k}$.

The previous theorems show that the number of maps left fixed by a graph automorphism depends considerably on the cycle type of the automorphism. We are led to define the *imbedding sum* of a graph G, denoted $Z(G)$, as the polynomial whose terms are the cycle-type monomials. Each automorphism γ whose cycle type corresponds to the monomial $\prod s_k^{j_k}$ contributes $|F(\gamma)|$ to the coefficient of this term in the polynomial. Formally, we have the following.

$$Z(G) = \frac{1}{|\Gamma(G)|} \sum_{\gamma \in \Gamma(G)} |F(\gamma)|\, s(\gamma). \tag{1}$$

We remark that this polynomial is no more difficult to determine using the counting theorems than the number of unlabeled imbeddings of the given graph. To illustrate, the number of unlabeled imbeddings of the complete graph K_n as reported in [8] is

$$\sum_{d|n} \frac{(n-2)!^{n/d}}{d^{n/d}(n/d)!} + \sum_{\substack{d|(n-1)\\ d\neq 1}} \frac{\phi(d)(n-2)!^{(n-1)/d}}{n-1}.$$

The imbedding sum is found to be

$$Z(K_n) = \sum_{d|n} \frac{(n-2)!^{n/d}}{d^{n/d}(n/d)!} s_d^{n/d} + \sum_{\substack{d|(n-1)\\ d\neq 1}} \frac{\phi(d)(n-2)!^{(n-1)/d}}{n-1} s_1 s_d^{(n-1)/d}. \tag{2}$$

The next theorem together with the counting theorems already presented are the main tools for the applications which follow. It is an immediate application of Redfield's lemma [10]. The notation used is that of Pólya [9]. If Γ is a permutation group, then the *cycle index* of Γ is the polynomial

$$Z(\Gamma) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} s(\gamma). \tag{3}$$

The notational similarity between the imbedding sum of a graph and the cycle index of a permutation group is justified with the following theorem, the main theorem of this article.

Theorem 5 (Decomposition Theorem). *$Z(G) = \sum Z(\Gamma_M(G))$, where the sum is over the set of unlabeled imbeddings of the graph G.*

Proof. Redfield's lemma asserts that if the group Γ acts on a set (the set of maps) and s is a function from Γ to a ring containing the rationals which is constant on conjugacy classes, then

$$\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} |F(\gamma)|\, s(\gamma) = \sum \frac{1}{|\Gamma_M|} \sum_{\gamma \in \Gamma_M} s(\gamma),$$

where the sum on the righthand side is over a complete set of orbit representatives (the unlabeled imbeddings). Taking the function s as the cycle-type monomial we observe that the lefthand side is $Z(G)$ and the righthand side is $\sum Z(\Gamma_M(G))$. □

3 Pólya Enumeration

The most immediate application of the imbedding sum is its evaluation at $s_k = 1$ for all k which is denoted as $Z(G;1)$. In this case, we have the number of unlabeled imbeddings of the graph. For example, $Z(K_4;1) = 3$, giving three unlabeled imbeddings of K_4. One of these is on the sphere and the other two are on the torus.

Suppose instead we have a function which assigns to each vertex of the graph an element of the set Y (say, for example, a set of colors). Moreover, suppose each element of Y is provided with an n-tuple of nonnegative integer weights. Each assignment is called a *figure*. The *figure counting series* $f(w_1, w_2, \ldots, w_n)$ records as the coefficient of $w_1^{p_1} w_2^{p_2} \cdots w_n^{p_n}$ the number of members of the set Y that have the n-tuple of weights $(p_1, p_2, \ldots, p_n)$.

Given a particular labeled imbedding M of the graph G and an assignment of members of Y to the vertices of G , we describe the assignments by the number of vertices whose assigned members from the set Y have weight $(p_1, p_2, \ldots, p_n)$. The map-automorphism group Γ_M then acts on these assignments in a natural way to yield equivalence classes of assignments. The classes are called *configurations* and are enumerated by weight in the *configuration counting series*.

The Pólya Enumeration Theorem [9] describes how this series can be determined from the cycle index $Z(\Gamma_M)$ and the figure counting series f. The substitution $s_k^{j_k} \rightarrow (f(w_1^k, w_2^k, \ldots, w_n^k))^{j_k}$ in the cycle index is denoted $Z(\Gamma_M; f(w_1, w_2, \ldots, w_n))$ and was shown by Pólya to provide the configuration counting series (see Harary and Palmer [5] for an excellent exposition of the theory). In summary, we having the following application of Pólya's theory.

Theorem 6. *If $f(w_1, w_2, \ldots, w_n)$ is the figure counting series for Y, then the series which enumerates by weight the number of configurations of the map M is $Z(\Gamma_M; f(w_1, w_2, \ldots, w_n))$.*

If we sum these series over a complete set of congruence class representative maps (the unlabeled imbeddings), then we obtain a series which enumerates by weight the quantity of configurations among the unlabeled imbeddings of the graphs. Moreover, from the Decomposition Theorem (5), this sum is the imbedding sum evaluated in the above manner. This evaluation is denoted as $Z(G; f(w_1, w_2, \ldots, w_n))$. We present the following theorem.

Theorem 7. *If $f(w_1, w_2, \ldots, w_n)$ is the figure counting series for Y, then the series which enumerates by weight the number of configurations of the graph G among its unlabeled imbeddings is $Z(G; f(w_1, w_2, \ldots, w_n))$.*

As an example, if we assign to each vertex of G a color from the set $Y = \{\text{black}, \text{white}\}$ whose members have weight (1,0) and (0,1), respectively, then the figure counting series is $f(b, w) = b^1 w^0 + b^0 w^1 = b + w$. In the case that G is the complete graph on four vertices we have from Equation 2

$$Z(K_4) = \frac{1}{24}(16s_1^4 + 32s_1s_3 + 12s_2^2 + 12s_4) \qquad \text{and}$$

$$Z(K_4; b+w) = \frac{1}{24}\left(16(b+w)^4 + 32(b+w)(b^3+w^3) + 12(b^2+w^2)^2 + 12(b^4+w^4)\right)$$

$$= \quad 3b^4 + 4b^3w + 5b^2w^2 + 4bw^3 + 3w^4.$$

Evidently there are five different ways to color the vertices in the three unlabeled imbeddings of K_4 so that two vertices receive the color black and two vertices receive the color white (the coefficient 5 of the term b^2w^2). These five colored unlabeled imbeddings are shown in Figure 1.

Observing that the two black vertices determine a unique edge in K_4 we have counted the number of different ways to root its unlabeled imbeddings at an edge. This technique can be used to determine the edge-rooted imbeddings of any complete graph, if desired.

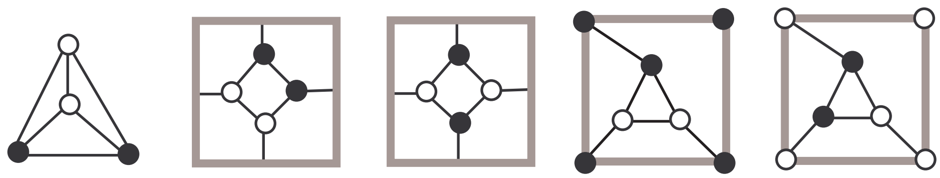

Figure 1: The different ways to color two vertices black among the unlabeled imbeddings of K_4. The last four imbeddings are on the torus.

In how many ways can the unlabeled imbeddings of K_n be rooted at a vertex? This question is answered by extracting the coefficient of b^1w^{n-1} in the configuration counting series $Z(K_n; b+w)$. This series is obtained from Equation 2 as

$$\begin{aligned} Z(K_n; b+w) &= \sum_{d|n} \frac{(n-2)!^{n/d}}{d^{n/d}(n/d)!}(b^d + w^d)^{n/d} \\ &+ \sum_{\substack{d|(n-1)\\ d\neq 1}} \frac{\phi(d)(n-2)!^{(n-1)/d}}{n-1}(b+w)(b^d+w^d)^{(n-1)/d}. \end{aligned} \tag{4}$$

Extracting the coefficient of b^1w^{n-1} and simplifying we have that the number of ways to root the unlabeled imbeddings of K_n at a single vertex is

$$\frac{1}{n-1}\sum_{d|n-1} \phi(d)(n-2)!^{(n-1)/d}. \tag{5}$$

Curiously, this resembles the cycle index of the cyclic group of order $n-1$. That is, if C_{n-1} is the cyclic permutation group of order $n-1$ and degree $n-1$, then

$$Z(C_{n-1}) = \frac{1}{n-1}\sum_{d|n-1} \phi(d)s_d^{(n-1)/d},$$

so that

$$Z(C_{n-1}; (n-2)!) = \frac{1}{n-1} \sum_{d|n-1} \phi(d)(n-2)!^{(n-1)/d}. \tag{6}$$

This result will become less surprising after we explore vertex-rooted graphs in detail in Section 4

4 The enumeration of graph imbeddings by their symmetries

4.1 Introduction

For many graphs the Decomposition Theorem (5) enables us to determine precisely how many imbeddings (both labeled and unlabeled) there are that have a particular map-automorphism group. To see this, consider the complete graph K_4 which has 16 labeled imbeddings and 3 unlabeled imbeddings.

One of the unlabeled imbeddings occurs on the sphere with map-automorphism group A_4 (the alternating group of order 12 represented as a permutation group of degree 4). The index of this group in the graph automorphism group is two so by Theorem 1 there are two labeled imbeddings in this congruence class.

The two other unlabeled imbeddings occur on the torus. One of them has as map-automorphism group the cyclic group of order 4, C_4 (represented as a permutation group of degree 4). The other toroidal imbedding has the cyclic group of order 3, $E_1 \times C_3$ (also represented as a permutation group of degree 4 where E_1 is the identity permutation group of degree 1). There are 6 labeled imbeddings in the first congruence class and 8 in the other.

The cycle indexes of these groups and the imbedding sum of K_4 satisfy the Decomposition Theorem (5) as indicated below.

$$\begin{array}{llllllll}
Z(A_4) & = & \frac{1}{12} & (s_1^4 & +8s_1s_3 & +3s_2^2 & &) \\
Z(C_4) & = & \frac{1}{4} & (s_1^4 & & +s_2^2 & +2s_4 &) \\
Z(E_1 \times C_3) & = & \frac{1}{3} & (s_1^4 & +2s_1s_3 & & &) \\
\hline
Z(K_4) & = & \frac{1}{24} & (16s_1^4 & +32s_1s_3 & +12s_2^2 & +12s_4 &)
\end{array}$$

Thus, $Z(K_4)$ decomposes as the sum of the cycle indexes $Z(A_4)$, $Z(C_4)$, and $Z(E_1 \times C_3)$. It is not difficult to show that no other decomposition of $Z(K_4)$ is possible. That is, if $Z(K_4) = \sum i_k Z(\Gamma_k)$, where the i_k are nonnegative integers and the Γ_k are permutation groups of degree 4 contained in $\Gamma(K_4)$, then the only solution is that given above.

Once again applying the Pólya Enumeration Theory of Section 3 with the figure counting series $b + w$ we have

$$Z(K_4; b+w) = Z(A_4; b+w) + Z(C_4; b+w) + Z(E_1 \times C_3; b+w). \tag{7}$$

Expanding each term on the righthand side and extracting the coefficients of b^2w^2 with the operator $[b^2w^2]$ we have

$$\begin{aligned}[b^2w^2]Z(K_4;b+w) &= [b^2w^2]Z(A_4;b+w)+[b^2w^2]Z(C_4;b+w)+[b^2w^2]Z(E_1\times C_3;b+w)\\ &= 1+2+2.\end{aligned}$$

Thus, among the five unlabeled imbeddings of K_4 with two black and two white vertices, one has symmetry group A_4, two have C_4, and two have $E_1 \times C_3$. Another glance at Figure 1 confirms this result.

It is unfortunate that uniqueness of the decomposition is not the rule. However, for some graphs many permutation groups can be eliminated from consideration as map-automorphism groups. In some cases then, additional information about the graph eliminates the ambiguity in the decomposition. We present some theorems which are useful in this regard. All but the last of these theorems are well known and their proofs may be found in White [12] or Biggs and White [2], for example.

Theorem 8. *If a map-automorphism group fixes two adjacent vertices, then it is the identity automorphism.*

Thus, it is not possible for the dihedral group D_4 to be a map-automorphism group of K_4 since as a permutation subgroup of $\Gamma(K_4)$ it would contain a permutation of cycle-type $s_1^2s_2$. This implies that the permutation fixes two vertices which are necessarily adjacent in K_4.

The next theorem will be used extensively to decompose the imbedding sum of wheel graphs, bouquets of loops, and vertex-rooted graphs.

Theorem 9. *If each member of a map-automorphism group Γ_M fixes the same vertex v, then Γ_M is a cyclic permutation group. Moreover, if the neighborhood of v has cardinality degree(v), then a generator of Γ_M restricted to the neighborhood of v is d-regular for some divisor d of degree(v). This implies that each member of Γ_M is a regular permutation when restricted to the neighborhood of v.*

Another useful result for excluding groups as map-automorphism groups is the following theorem which limits their order.

Theorem 10. *If a graph has e edges, then the order of a map-automorphism group must be a divisor of $2e$. Moreover, if the order of some map-automorphism group attains the value $2e$, then the group acts transitively on the vertices, edges, and regions of the imbedded graph.*

In most of the applications to follow we will ensure a unique decomposition of the imbedding sum by restricting the investigation to graphs whose map-automorphism groups are all cyclic (invoking Theorem 9). In these cases, the number of unlabeled imbeddings with a particular cyclic symmetry can be explicitly found. The cycle indexes of these cyclic groups have the form $\frac{1}{d}\sum_{k|d}\phi(k)s_k^{n/k}$ (or $s_1s_k^{n/k}$) (see [5]).

Theorem 11. *If the map-automorphism groups of a graph are cyclic, then its imbedding sum uniquely decomposes as the sum of cyclic cycle indexes. Let G_n be such a graph, the size of which is measured by the integer parameter n, and let i_d be the number*

of unlabeled imbeddings whose map-automorphism group is cyclic of order d. Suppose further that there is a function, f, of n and d alone such that the imbedding sum of G_n satisfies

$$Z(G_n) = \sum_{d|n} f(n,d) s_d^{n/d} \quad \textit{(or } s_1 s_d^{n/d}\textit{), then}$$

$$i_d = d \sum_{\substack{k \\ d|k|n}} \frac{f(n,k)}{\phi(k)} \mu(k/d), \quad \textit{where } \mu \textit{ is the Möbius function.} \tag{8}$$

Proof. From the Decomposition Theorem (5) we have

$$\sum_{d|n} f(n,d) s_d^{n/d} = \sum_{k|n} i_k \frac{1}{k} \sum_{d|k} \phi(d) s_d^{n/d} \qquad (\text{or } s_1 s_d^{n/d}).$$

There is a unique solution to this system of equations in the unknowns i_k obtained by comparing coefficients of s_n (or $s_1 s_n$) and then, in turn, the coefficients of $s_d^{n/d}$ (or $s_1 s_d^{n/d}$) for each successive divisor d of n beginning with the largest. Uniqueness of the solution was actually shown to be the case in Redfield's original article [10]. Extracting the coefficient of $s_d^{n/d}$ (or $s_1 s_d^{n/d}$) yields

$$f(n,d) = \sum_{\substack{k \\ d|k|n}} i_k \frac{\phi(d)}{k} \quad \text{or} \quad \frac{f(n,d)}{\phi(d)} = \sum_{\substack{k \\ d|k|n}} \frac{i_k}{k}.$$

The latter system of equations (one equation for each divisor d of n) can be solved explicitly for i_d/d by Möbius inversion. The details of this particular inversion problem can be found in Hall [4]. □

4.2 Wheel graphs.

We apply the Decomposition Theorem (5) to a class of graphs where the map-automorphism groups are known. This is the class of wheel graphs, W_{n+1}, described as a cycle of n vertices each joined to a central vertex. The counting theorems were applied to this class of graphs in [8] to obtain the number of unlabeled imbeddings for each of its members. From this information it is easy to determine that the imbedding sum for wheels on five or more vertices is

$$Z(W_{n+1}) = \begin{cases} \frac{1}{2n} \sum_{d|n} \frac{\phi^2(d)}{d} (2d)^{n/d} (\frac{n}{d} - 1)! s_1 s_d^{n/d} & \text{if } n \text{ is odd,} \\ \frac{1}{2n} \sum_{d|n} \frac{\phi^2(d)}{d} (2d)^{n/d} (\frac{n}{d} - 1)! s_1 s_d^{n/d} + 2^{n-3} (\frac{n}{2} - 1)! s_1 s_2^{n/2} & \text{if } n \text{ is even.} \end{cases} \tag{9}$$

To characterize the map-automorphism groups of the wheel graphs we note that each graph automorphism must fix the central vertex (provided that $n \geq 4$). Thus, if M is a map of W_{n+1} with map-automorphism group Γ_M, then an immediate result of Theorem 9 is that Γ_M must be $E_1 \times C_d[E_{n/d}]$ for some divisor d of n, where E_1 is the identity permutation group of degree 1 and $C_d[E_{n/d}]$ is the wreath product of the cyclic

Table 1: The number of unlabeled imbeddings of the wheel W_{n+1} which have d-fold symmetry.

n / d	4	5	6	7	8
1	9	76	617	6582	80399
2	5	0	42	0	479
3	0	0	5	0	0
4	2	0	0	0	4
5		4	0	0	0
6			2	0	0
7				6	0
8					4
Total	16	80	666	6588	80886

permutation group of order d and degree d about the identity permutation group of degree n/d. The cycle index of this group is $\frac{1}{d}\sum_{k|d}\phi(k)s_1 s_k^{n/k}$.

Let i_d be the number of unlabeled imbeddings of W_{n+1} which have map-automorphism group $C_d[E_{n/d}]$. We say that these imbeddings are *d-fold symmetric*. Applying the Decomposition Theorem (5) we have that

$$Z(W_{n+1}) = \sum_{d|n} i_d \frac{1}{d} \sum_{k|d} \phi(k) s_1 s_k^{n/k}. \tag{10}$$

The explicit solution obtained by using Theorem 11 is given below.

Theorem 12. *The number i_d of unlabeled imbeddings of the wheel W_{n+1} which are d-fold symmetric is given by*

$$i_d = d \sum_{\substack{k \\ d|k|n}} \frac{f(n,k)}{\phi(k)} \mu(k/d), \quad \textit{where}$$

$$f(n,k) = \begin{cases} \frac{\phi^2(k)}{2nk}(2k)^{n/k}(\frac{n}{k}-1)! & \textit{if } k \neq 2, \\ \frac{\phi^2(k)}{2nk}(2k)^{n/k}(\frac{n}{k}-1)! + 2^{n-3}(\frac{n}{2}-1)! & \textit{if } k = 2. \end{cases}$$

The number of unlabeled imbeddings of W_{n+1} which are d-fold symmetric is given in Table 1 for the first few values of n. From this information and Theorem 1 we are able to calculate the number of labeled imbeddings of W_{n+1} which have d-fold symmetry. This information is tabulated in Table 2.

4.3 Bouquets

Another graph whose map-automorphism groups are cyclic is the bouquet of n loops, B_n (represented as a simple graph by introducing two vertices upon each loop). For this

Table 2: The number of labeled imbeddings of the wheel W_{n+1} which have d-fold symmetry.

n d	4	5	6	7	8
1	72	760	7404	92148	1286384
2	20	0	252	0	3832
3	0	0	20	0	0
4	4	0	0	0	16
5		8	0	0	0
6			4	0	0
7				12	0
8					8
Total	96	768	7680	92160	1290240

graph, the graph-automorphism group is the permutation group $E_1 \times S_n[C_2]$ of order $2^n n!$ and degree $2n+1$, where S_n is the symmetric permutation group.

The imbedding sum is (Rieper [11])

$$Z(B_n) = \frac{1}{2n}\sum_{d|n} \frac{\phi(d)(2n/d)!d^{n/d}}{2^{n/d}(n/d)!} s_1 s_d^{2n/d} + \frac{1}{2n}\sum_{d|n}\sum_{m=0}^{\lfloor\frac{n-1}{2d}\rfloor} \frac{\phi(2d)(n/d)!d^m}{(\frac{n}{d}-2m)!m!} s_1 s_{2d}^{n/d}. \quad (11)$$

A consequence of Theorem 9 is that each map-automorphism group of the bouquet $B_n, n > 1$, is one of the permutation groups $E_1 \times C_d[E_{2n/d}]$ for some divisor d of $2n$. A map with this map-automorphism group is again said to be d-fold symmetric.

Performing an analysis like that for the wheel graphs we determine the number of unlabeled and labeled imbeddings of the bouquet which are d-fold symmetric. The results of the calculations are given in the theorem below and are tabulated in Table 3 and Table 4.

Theorem 13. *The number i_d of unlabeled imbeddings of the bouquet B_n which are d-fold symmetric is given by*

$$i_d = d \sum_{\substack{k \\ d|k|n}} f(n,k)\mu(k/d), \quad \textit{where}$$

$$f(n,k) = \begin{cases} \frac{(2n/k)!(k/2)^{n/k}}{2n(n/k)!} & \textit{if } k \textit{ is odd,} \\ \frac{(2n/k)!(k/2)^{n/k}}{2n(n/k)!} + \frac{1}{2n}\sum_{m=0}^{\lfloor\frac{n-1}{k}\rfloor} \frac{(2n/k)!(k/2)^m}{(2n/k-2m)!m!} & \textit{if } k \textit{ is even.} \end{cases}$$

4.4 The directed bouquet

We illustrate here that the counting and decomposition theorems can be used to enumerate graph imbeddings for graphs that have additional structure such as a directed

Table 3: The number of unlabeled imbeddings of the bouquet B_n which have d-fold symmetry.

n d	1	2	3	4	5	6
1	0	0	1	10	86	837
2	1	1	2	5	16	52
3		0	1	0	0	5
4		1	0	2	0	4
5			0	0	2	0
6			1	0	0	3
7				0	0	0
8				1	0	0
9					0	0
10					1	0
11						0
12						1
Total	1	2	5	18	102	902

Table 4: The number of labeled imbeddings of the bouquet B_n which have d-fold symmetry.

n d	1	2	3	4	5	6
1	0	0	48	3840	330240	38568960
2	1	4	48	960	30720	1198080
3		0	16	0	0	76800
4		2	0	192	0	46080
5			0	0	1536	0
6			8	0	0	23040
7				0	0	0
8				48	0	0
9					0	0
10					384	0
11						0
12						3840
Total	1	6	120	5040	362880	39916800

graph. We choose the directed bouquet of n loops, $\overrightarrow{B_n}$, which is the bouquet with each edge (loop) given an orientation (indicated by placing an arrow along each loop).

The counting theorems listed in Section 3 arose from an analysis of the action of the graph-automorphism group on the *vertices* of the graph but the graph-automorphism group of the directed bouquet acts on its set of directed edges. We overcame a similar problem for the undirected bouquet by twice subdividing each loop to yield a simple graph. This same technique is used here except that one of the two vertices introduced upon an edge is colored black and the other white. We agree then, that the orientation of the edge proceeds from the black to the white vertex. The graph automorphisms must map a black vertex to a black vertex, a white vertex to another white vertex, and the original central vertex to itself.

With this model we can use the counting and decomposition theorems as they were presented with the following modification. We record the cycle type of any automorphism with respect to its action on the directed edges instead of its action on the vertices.

The graph-automorphism group of $\overrightarrow{B_n}$ is the symmetric group S_n. Theorem 4 implies that we only need to consider the permutations in S_n that are d-regular for some divisor d of n. Let γ be one of the $\frac{n!}{d^{n/d}(n/d)!}$ d-regular permutations in S_n. Let v be the central vertex, then $l(v,\gamma) = 1$ so the fixed set at v satisfies

$$|F_v(\gamma)| = \phi(d)\left(\frac{2n}{d} - 1\right)! d^{\frac{2n}{d}-1}.$$

If u is a noncentral vertex (either black or white), then it has two neighbors one of which is the central vertex. Thus, $\gamma^{l(u,\gamma)}$ is 1-regular on the neighborhood of u so by Theorem 4 the fixed set at u has cardinality 1.

From this information and Theorem 2 the imbedding sum of $\overrightarrow{B_n}$ is given by

$$\begin{aligned} Z(\overrightarrow{B_n}) &= \frac{1}{n!}\sum_{d|n} \frac{n!}{d^{n/d}(n/d)!}\phi(d)\left(\frac{2n}{d} - 1\right)! d^{\frac{2n}{d}-1} s_d^{n/d} \qquad (12) \\ &= \sum_{d|n} \frac{\phi(d)(2n/d)! d^{n/d}}{2n(n/d)!} s_d^{n/d}. \end{aligned}$$

The map-automorphism groups are once again forced to be cyclic by the presence of the central vertex. Each must be $C_d[E_{n/d}]$ for some divisor d of n with cycle index $\frac{1}{d}\sum_{k|d}\phi(k)s_k^{n/k}$. Letting i_d denote the number of unlabeled imbeddings with this map-automorphism group we have from Theorem (5) that

$$Z(\overrightarrow{B_n}) = \sum_{d|n} i_d \frac{1}{d}\sum_{k|d}\phi(k)s_k^{n/k}. \qquad (13)$$

Equating the righthand sides of Equation 12 and Equation 13 and using Theorem 11 we have

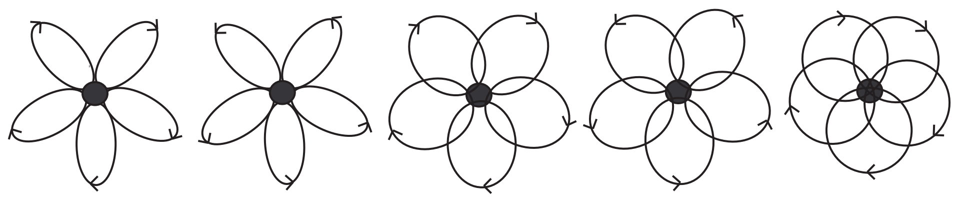

Figure 2: The five different imbeddings of the directed bouquet $\vec{B_5}$ of maximum symmetry represented as overlap graphs. The first two drawings represent planar imbeddings and the last three represent toroidal imbeddings.

Theorem 14. *The number i_d of unlabeled imbeddings of the directed bouquet $\overrightarrow{B_n}$ which are d-fold symmetric is given by*

$$i_d = \frac{d}{2n} \sum_{\substack{k \\ d|k|n}} \mu(k/d) \frac{(2n/k)!k^{n/k}}{(n/k)!}. \tag{14}$$

Some results are tabulated in Table 5 and Table 6. Note that the number of unlabeled imbeddings of $\overrightarrow{B_n}$ with the largest map-automorphism group, C_n, is n. This is easily confirmed from Equation 14 above.

In Figure 2 we depict the five unlabeled imbeddings of $\overrightarrow{B_5}$ as *overlap* graphs. The overlap graphs are obtained from the surface imbeddings by projection onto the plane. These overlap graphs are very useful constructs and in our case capture the 5-fold symmetry of the imbeddings very nicely. However, these drawings can be misleading. Why isn't there a sixth drawing paired with the fifth by a reversal of the directed edges as is the case for the first two pairs? We leave the answer to the curious reader.

4.5 Vertex-rooted graphs

As mentioned previously, if we desire to enumerate the imbeddings of a graph by their map-automorphism groups, then we need more information beyond its imbedding sum. Information about its map-automorphism groups must be available because the decomposition of the imbedding sum given by the Decomposition Theorem (5) is not, in general, unique. In the examples given so far, the map-automorphism groups were all cyclic and in this case the decomposition of the imbedding sum can be found.

If the map-automorphism groups of the graph are not all cyclic, then we have two choices. Either additional information must be made available or we must alter the graph to force the map-automorphism groups to be cyclic. The latter can be accomplished by rooting the graph at a vertex. In general, there are numerous choices for the root but we denote any one of them as G^*. Every automorphism of the rooted graph

Table 5: The number of unlabeled imbeddings of the directed bouquet $\vec{B_n}$ which have d-fold symmetry.

n d	1	2	3	4	5	6	7	8
1	1	3	20	204	3023	55352	1235519	32430720
2		2	0	10	0	158	0	3336
3			3	0	0	24	0	0
4				4	0	0	0	44
5					5	0	0	0
6						6	0	0
7							7	0
8								8
Total	1	5	23	218	3028	55540	1235526	32434108

Table 6: The number of labeled imbeddings of the directed bouquet $\vec{B_n}$ which have d-fold symmetry.

n d	1	2	3	4	5	6
1	1	6	120	4896	362760	39853440
2		2	0	120	0	56880
3			6	0	0	5760
4				24	0	0
5					120	0
6						720
Total	1	8	126	5040	362880	39916800

must fix the root and, hence, by Theorem 9, each map-automorphism group is cyclic. Information about the imbedding symmetries of the rooted graph is then available as in the previous cases.

As an example, we consider the vertex-rooted complete graph K_n^*. The automorphism group of this graph is $E_1 \times S_{n-1}$. Since each automorphism must fix the rooted vertex, Theorem 8 implies that no non-identity map-automorphism can fix another vertex and Theorem 9 implies we need only consider those automorphisms of cycle-type $s_1 s_d^{(n-1)/d}$ for some divisor d of $n-1$. There are $(n-1)!/(d^{(n-1)/d} \cdot ((n-1)/d)!)$ such permutations in $E_1 \times S_{n-1}$. Let γ be one of these permutations and let u be a vertex other than the root. Then $l(u,\gamma) = d$ so that $\gamma^{l(u,\gamma)}$ is the identity permutation. Applying Theorem 4, the fixed set at u for γ satisfies $\left|F_u(\gamma^{l(u,\gamma)})\right| = (n-2)!$ since $\gamma^{l(u,\gamma)}$ is 1-regular (it is the identity). There are $\frac{n-1}{d}$ orbit representatives like the vertex u under the action of γ so their contribution to the cardinality of the fixed set of γ is the factor $(n-2)!^{(n-1)/d}$ (see Theorem 3).

If v is the rooted vertex, then $l(v,\gamma) = 1$ and we have from Theorem 4 that

$$\left|F_v(\gamma^{l(v,\gamma)})\right| = \phi(d)\left(\frac{n-1}{d} - 1\right)! d^{\frac{n-1}{d}-1}.$$

Thus, the fixed set for γ satisfies

$$|F(\gamma)| = \phi(d)\left(\frac{n-1}{d} - 1\right)! d^{\frac{n-1}{d}-1}(n-2)!^{\frac{n-1}{d}}.$$

Summing over all the d-regular permutations we find after simplifying that the imbedding sum is

$$Z(K_n^*) = \frac{1}{n-1}\sum_{d|(n-1)} \phi(d)(n-2)!^{(n-1)/d} s_1 s_d^{(n-1)/d}. \tag{15}$$

This is in agreement with Equation 5 derived by applying the Pólya Enumeration Theory to the imbedding sum of the unrooted complete graph.

Each of the map-automorphism groups of K_n^* must be $E_1 \times C_d[E_{(n-1)/d}]$ for some divisor d of $n-1$, in which case the corresponding map is d-fold symmetric. The imbedding sum uniquely decomposes as a sum of these groups. Applying Theorem 11 we have

Theorem 15. *The number i_d of unlabeled imbeddings of the vertex-rooted complete graph K_n^* which are d-fold symmetric is given by*

$$i_d = d \sum_{\substack{k \\ d|k|n-1}} \mu(k/d)\frac{(n-2)!^{(n-1)/k}}{n-1}.$$

The numbers of unlabeled and labeled imbeddings of K_n^* which are d-fold symmetric are given in Table 7 and Table 8, respectively.

Of special interest is the number of imbeddings which have the largest possible map-automorphism group, $E_1 \times C_{n-1}$. The number of unlabeled imbeddings is $i_n = (n-2)!$. There are $(n-2)!$ different labeled imbeddings in each of these congruence classes. In the case n equals 4, two labeled imbeddings on the sphere and two on the torus have 3-fold symmetry.

Table 7: The number of unlabeled imbeddings of the vertex-rooted complete graph K_n^* which have d-fold symmetry.

n / d	1	2	3	4	5	6	7
1	1	1	0	2	315	1592520	497662709620
2		0	1	0	15	0	575960
3			0	2	0	0	7140
4				0	6	0	0
5					0	24	0
6						0	120
Total	1	1	1	4	336	1592548	497663292840

Table 8: The number of labeled imbeddings of the vertex-rooted complete graph K_n^* which have d-fold symmetry.

n / d	1	2	3	4	5	6
1	1	1	0	12	7560	191102400
2		0	1	0	180	0
3			0	4	0	0
4				0	36	0
5					0	576
6						0
Total	1	1	1	16	7776	191102976

4.6 The complete graph K_5.

The alternative to rooting the graph at a vertex is additional information about its map-automorphism groups. We illustrate by decomposing the imbedding sum of the complete graph K_5. There are $(4!)^5 = 7776$ different labeled imbeddings and $Z(K_5;1) = 78$ unlabeled imbeddings. The imbedding sum (Equation 2) is

$$Z(K_5) = \frac{1}{120}(7776s_1^5 + 1080s_1s_2^2 + 360s_1s_4 + 144s_5). \tag{16}$$

There are numerous decompositions of this sum as the sum of cycle indexes of permutation groups. However, from Theorem 10 we know that each map-automorphism group of K_5 must have an order that is a divisor of 20. Thus, for example, we can exclude the groups $E_1 \times C_2[C_2]$, $E_2 \times C_3$, and $C_2 \times C_3$. Theorem 8 implies that we can also exclude $E_1 \times C_2 \times C_2$ and others which have a nonidentity element that fixes two adjacent vertices.

In addition, it is known that a complete graph with a prime power number of vertices has a map-automorphism group of maximum order and that each is a Frobenius group (Biggs [1]). For K_5 we denote this group as F and report that its cycle index is $Z(F) = \frac{1}{20}(s_1^5 + 5s_1s_2^2 + 10s_1s_4 + 4s_5)$.

The other possible map-automorphism groups are the dihedral group D_5, the identity group E_5, and the cyclic groups C_5, $E_1 \times C_4$, and $E_1 \times C_2[E_2]$. We let i_F be the number of unlabeled imbeddings with map-automorphism group the Frobenius group, i_5, i_4, and i_2 the quantity with cyclic map-automorphism groups C_5, $E_1 \times C_4$, and $E_1 \times C_2[E_2]$, respectively, i_D the number with dihedral symmetry, and i_1 the number of asymmetric unlabeled imbeddings (symmetry group E_5). We are led to solve the equation

$$\begin{array}{rlllllll}
\multicolumn{8}{l}{\frac{1}{120}(7776s_1^5 + 1080s_1s_2^2 + 360s_1s_4 + 144s_5) =} \\
 & i_F \cdot \frac{1}{20}(& s_1^5 & +5s_1s_2^2 & +10s_1s_4 & +4s_5 &) \\
+ & i_5 \cdot \frac{1}{5}(& s_1^5 & & & +4s_5 &) \\
+ & i_4 \cdot \frac{1}{4}(& s_1^5 & +s_1s_2^2 & +2s_1s_4 & &) \\
+ & i_2 \cdot \frac{1}{2}(& s_1^5 & +s_1s_2^2 & & &) \\
+ & i_D \cdot \frac{1}{10}(& s_1^5 & +5s_1s_2^2 & & +4s_5 &) \\
+ & i_1 \cdot \frac{1}{1}(& s_1^5 & & & &).
\end{array} \tag{17}$$

By comparing coefficients there are four different solutions to the above. One of the solutions gives i_5 equal to one, the other three solutions give i_5 equal to zero. Thus, if we can construct a map of K_5 which has C_5 as its map-automorphism group, then we would be done. This we do now.

Label the vertices of K_5 with the integers 1 through 5 and let γ be the automorphism $(1, 2, 3, 4, 5)$. Choose the rotation at vertex 1 to be the cyclic permutation $(2, 3, 4, 5)$ and let γ act on this permutation repeatedly by conjugation to produce the rotations at the other vertices. The rotation system is then $\rho_1 = (2, 3, 4, 5)$, $\rho_2 = (3, 4, 5, 1)$, $\rho_3 = (4, 5, 1, 2)$, $\rho_4 = (5, 1, 2, 3)$, and $\rho_5 = (1, 2, 3, 4)$.

The map-automorphism group of the corresponding imbedding then has C_5 as a subgroup. From the rotation system we find that the imbedding has two 5-sided regions and one 10-sided region (see Gross and Tucker [3] for the algorithm which produces the region sizes from the rotation system). Thus, we know from Theorem 10 that the map-automorphism group can not be the Frobenius group F which requires all regions to be of the same size. The group is then either C_5 or the dihedral group D_5. If it is the latter, then the automorphism $\alpha = (1)(2,5)(3,4)$ must fix the rotation system above. However α takes $\rho_1 = (2,3,4,5)$ to $(5,4,3,2)$ by conjugation and this is not $\rho_{\alpha 1} = \rho_1$.

Thus, we have produced an imbedding of K_5 which has map-automorphism group C_5. The decomposition of the imbedding sum is now uniquely determined to be $i_F = 2$, $i_5 = 1$, $i_D = 0$, $i_4 = 4$, $i_2 = 15$ and $i_1 = 56$.

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